GUARDED MODAL FIXED-POINT EQUATIONS AND CONVERSE WELL-FOUNDEDNESS

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Abstract. We study semantic fixed-point equations of the form $p \leftrightarrow A(p)$ in propositional modal logic, where every occurrence of p in A is guarded by at least one modal operator. On every converse well-founded Kripke frame, such an equation admits a unique semantic solution without any monotonicity assumption on the induced operator (Theorem 5.1). This extends to finite systems of mutually recursive guarded equations, by a vectorial dependency lemma (Theorem 9.4).

The main conceptual contribution is a characterization (Theorem 6.4): a Kripke frame F is converse well-founded if and only if $p \leftrightarrow \Diamond p$ has a unique semantic solution under every valuation on F. This reduces a converse-well-foundedness condition—which is not first-order definable but monadic-second-order expressible—to a single propositional fixed-point uniqueness condition, without appeal to validity or to any proof system. Correspondingly, the least and greatest fixed points of the modal μ-calculus coincide on converse well-founded frames for every positive guarded formula (Corollary 8.3); this is a restricted special case of a known collapse result, included as an illustration.

The paper also records the structural property of *rank-locality* underlying the uniqueness theorem (Section 11), isolates its semantic content from its syntactic implementation via guardedness, and, in a final section of outlook (Section 10), formulates the more speculative program of *fixed-point uniqueness as a mode of frame characterization*: a monadic-second-order upper bound on the FPU-spectrum, a non-first-order witness in the polymodal setting, and open problems.

1. Introduction

Recursive definitions in modal logic appear in several familiar settings. Provability logic admits fixed-point phenomena for formulas modalized in the distinguished variable, captured syntactically by the de Jongh–Sambin theorem [6, 17]: for every formula $B(p)$ of GL in which p is modalized, there is a sentence ψ (not containing p) with $\mathrm{GL} \vdash B(\psi) \leftrightarrow \psi$ and $\mathrm{GL} \vdash (B(p) \leftrightarrow p) \to (p \leftrightarrow \psi)$. The modal μ-calculus [11, 7] introduces least and greatest fixed points $\mu p.A$ and $\nu p.A$ for positive (hence monotone) modal operators, and the Arnold–Niwiński collapse theorem [3, Theorem 4.2.6] shows that on well-founded transition systems these extremal fixed points coincide, making the μ-calculus alternation hierarchy collapse. In type theory and programming semantics, guarded recursion as introduced by Nakano [13] and formalised in the topos of trees by Birkedal et al. [4] yields unique fixed points for contractive endofunctors.

These three traditions — provability-logical, μ-calculus-theoretic, and categorical/topos-theoretic — approach the question of when semantic recursion is well-behaved from different angles. The first answers it with a syntactic theorem within a specific proof system; the second with least and greatest fixed-point operators over monotone functors; the third with a Banach-style contractiveness argument in a suitable category of presheaves.

The present paper isolates a *fourth* angle, intermediate between these and arguably simpler than any of them. We work in propositional modal logic with Kripke semantics; we require neither a proof system, nor monotonicity, nor a topos-theoretic ambient structure. We study

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equations

$$p \leftrightarrow A(p),$$

where every occurrence of p in A is guarded by at least one modal operator, and ask when such an equation admits a unique semantic solution over a given frame. Guardedness imposes a semantic delay: at a world w, the truth of $A(p)$ depends on the interpretation of p only at worlds strictly accessible from w. If the accessibility relation admits no infinite forward branch, this dependence unfolds along a well-founded rank and resolves into a unique solution by transfinite recursion (Theorem 5.1).

The characterization. The central result is a tight converse to the uniqueness theorem. Write $F \models \mathrm{FPU}(\Phi)$ when, for every valuation of the parameters on F, the equation $p \leftrightarrow \Phi$ has exactly one semantic solution.

> **Theorem 6.4.** For every Kripke frame $F = (W, R)$, the following are equivalent:
> (1) F is converse well-founded;
> (2) $F \models \mathrm{FPU}(A)$ for every A guarded in p;
> (3) $F \models \mathrm{FPU}(\Diamond p)$.

This reduces converse well-foundedness—a property not first-order definable but monadic-second-order expressible—to the semantic uniqueness of a single propositional modal equation.

The hole in the literature. To our knowledge, the characterization above does not appear in the existing literature in any of the three traditions we have listed, and cannot be obtained as a straightforward corollary of existing results. Concretely:

Relation to de Jongh–Sambin. Sambin's fixed-point theorem is a *syntactic* theorem in GL, which additionally assumes transitivity and irreflexivity of the frame; our Theorem 5.1 requires only converse well-foundedness, is purely semantic, and the unique solution need not be modally definable. The characterization (Theorem 6.4) has no syntactic analogue in that tradition: it characterises a frame property by the *number* of semantic solutions of an equation, not by validity of a modal formula.

Relation to Arnold–Niwiński. The μ/ν collapse on well-founded systems asserts $\mu p.A = \nu p.A$ for positive A; our Theorem 5.1 asserts $|\mathrm{Fix}(T_A)| = 1$ for guarded A. The first operates on monotone operators and compares extremal fixed points; the second operates without monotonicity and counts all fixed points. For non-monotone A (such as $\Box\neg p$), the μ/ν apparatus does not even apply, yet Theorem 5.1 delivers uniqueness (Proposition 8.2).

Relation to guarded recursion and step-indexing. Nakano's later modality and its topos-theoretic formalisation [13, 4] produce unique fixed points for contractive endofunctors on presheaf categories, via a Banach-style argument. The underlying principle — unique unfolding of delayed recursion over a well-founded structure — is morally the same as ours. Our contribution at this level is a Kripke-semantic reformulation with minimal and sharp hypotheses, made precise through the structural property of *rank-locality* (Proposition 11.3). We do not claim a formal reduction to the categorical theorems of [4]; whether such a reduction exists, via an embedding of converse well-founded Kripke frames into a suitable topos, remains open.

Relation to modal correspondence theory. Classical modal correspondence and Sahlqvist theory [5, 19] characterise frame properties by the *validity* of modal formulas. Our Theorem 6.4 characterises converse well-foundedness by the *uniqueness* of the solution set of a modal equation. This is a distinct mode of characterization, neither subsumed by first-order correspondence nor by Sahlqvist's algorithm.

The picture we put forward, therefore, is the following. Converse well-foundedness sits at the confluence of several well-studied settings: provability logic, μ-calculus, guarded recursion. In each of these settings there are fixed-point phenomena, all of them morally connected. Theorem 6.4 is a Kripke-semantic, purely propositional, equation-theoretic statement that sits in a space which — despite the richness of the surrounding traditions — had not been written down explicitly. It is a natural statement filling a specific gap in the literature.

Contributions. The contributions are organised on three levels.

Main result. The characterization theorem (Theorem 6.4): a Kripke frame is converse well-founded if and only if $p \leftrightarrow \Diamond p$ has a unique semantic solution under every valuation.

Supporting results. The uniqueness theorem for guarded equations on converse well-founded frames without monotonicity hypotheses (Theorem 5.1); the vectorial extension to finite systems with a complete dependency lemma (Theorem 9.4, Lemma 9.2); the isolation of rank-locality as the abstract semantic core, with a fixed-point theorem independent of guardedness (Theorem 11.2); a restricted collapse $\mu p.A = \nu p.A$ on CWF frames for positive guarded A, as a special case of Arnold–Niwiński (Corollary 8.3).

Extensions (more speculative). A monadic second-order upper bound on the FPU-spectrum (Theorem 10.11); a non-first-order witness in the bimodal setting (Theorem 10.13); a counterexample (Proposition 10.5) showing that not every guarded effective formula characterises CWF; and a conjecture on the unimodal landscape (Conjecture 10.18).

The final section (Section 10) formulates the more speculative program of *fixed-point uniqueness as a mode of frame characterization*, with open problems.

Organisation. Section 2 reviews related work and articulates the separation from existing traditions. Section 3 fixes notation. Section 4 proves the dependency lemma. Section 5 proves the uniqueness theorem. Section 6 proves the characterization. Section 7 gives unary examples. Section 8 articulates the precise distance from the μ/ν collapse. Section 9 develops the vectorial extension with complete proofs. Section 11 treats rank-locality. Section 10 is the outlook on FPU-characterization. Section 12 concludes with open problems.

2. Related Work

We situate the paper with respect to five strands of research. Each of them provides part of the surrounding context, and none of them subsumes Theorem 6.4.

Provability logic and the de Jongh–Sambin theorem. The classical fixed-point theorem of provability logic [6, 17] states that for every GL-formula $B(p)$ in which p is modalized, there exists a sentence ψ (not containing p) such that $\mathrm{GL} \vdash B(\psi) \leftrightarrow \psi$ and $\mathrm{GL} \vdash (B(p) \leftrightarrow p) \to (p \leftrightarrow \psi)$. Uniqueness of the fixed-point sentence up to GL-equivalence follows. The theorem was first proved by de Jongh (unpublished) and by Sambin independently [16, 17, 18]. Alternative proofs and refinements were given by Montagna and by Reidhaar-Olson [8, 14]; generalisations to other provability logics (in particular extensions with Sacchetti-type principles) appear in [15]; and algebraic/semantic approaches are developed in Visser's work on interpretability and fixed points [21].

The de Jongh–Sambin theorem, however elegant, differs from Theorem 6.4 along several axes: (i) it is a syntactic theorem within a specific proof system; (ii) it assumes both transitivity and irreflexivity of the frame (GL is sound and complete over finite transitive irreflexive trees); (iii) its output is a provably definable fixed-point *sentence*, not merely a semantic object. Our Theorem 5.1 requires only converse well-foundedness (no transitivity, no irreflexivity), operates purely semantically, and allows the unique solution to be a set not expressible by any modal formula. The characterization (Theorem 6.4) has no analogue in the de Jongh–Sambin tradition: that tradition characterises a *proof system*, not a frame property, and does so by syntactic definability rather than semantic uniqueness of solutions.

The modal μ-calculus. Kozen's propositional μ-calculus [11] introduces least and greatest fixed points for positive (monotone) modal operators. Standard expositions include [7, 3]. On well-founded transition systems the alternation hierarchy collapses, by [3, Theorem 4.2.6]; this generalises the observation that $\mu p.\Diamond p = \emptyset$ and $\nu p.\Diamond p$ equals the non-well-founded part of the relation, and that on CWF frames these coincide.

The logical content of Theorem 6.4 is distinct from this collapse in two ways. First, FPU quantifies over *all* fixed points, not only the extremal ones, and applies to non-monotone operators where μ and ν are not defined in the Knaster–Tarski sense (Proposition 8.2). Second, FPU is a condition on the number of solutions of a modal equation; the Arnold–Niwiński collapse is a statement about the equality of extremal fixed points of a monotone operator. conditions operate at different levels: $\mu = \nu$ implies a singleton Knaster–Tarski interval, but not uniqueness in $\mathcal{P}(W)$ tout court. The two conditions happen to coincide for positive A, but their formulations operate at different levels.

Guarded recursion and step-indexed semantics. Nakano [13] introduced a modality $\triangleright$ ("later") to give semantics to recursive type definitions. Birkedal, Møgelberg, Schwinghammer, and Støvring [4] formalise this in the topos of trees $\mathrm{Set}^{\omega^{\mathrm{op}}}$, where guarded functors have unique fixed points by a Banach-style argument. Step-indexed semantics, due to Appel and McAllester [1], treats the interpretation at stage n as depending only on stages $< n$, which is precisely the ω-indexed analogue of our rank-indexed construction.

The structural principle underlying Theorem 5.1 — delayed recursion over a well-founded structure yields a unique unfolding — is the same principle that animates these traditions. Our contribution at this level is *not* a new mathematical insight but a reformulation within Kripke semantics under minimal hypotheses, and the isolation of rank-locality (Proposition 11.3, Theorem 11.2) as the abstract semantic condition implied by guardedness. Whether Theorem 5.1 can be obtained as a formal corollary of the fixed-point theorem for contractive endofunctors on an appropriate topos, via a functor embedding of converse well-founded Kripke frames, is an interesting question that we leave open.

Modal correspondence theory. Classical modal correspondence theory, developed by van Benthem and successors [5, 19, 20], characterises frame properties by the *validity* of modal formulas: φ corresponds to a first-order sentence α on frames if $F \models \varphi \iff F \models \alpha$ for all frames F. Sahlqvist's theorem [5] provides an effective algorithm computing the first-order correspondent of a syntactically restricted class of modal formulas.

Theorem 6.4 characterises converse well-foundedness not by the validity of $p \leftrightarrow \Diamond p$ but by the *uniqueness* of the solution set of this equation. This is a different mode of characterization, neither reducible to first-order correspondence nor to Sahlqvist theory. The property characterised (converse well-foundedness) is not first-order definable; classical correspondence would therefore locate it in the monadic second-order fragment. Our theorem delivers a different handle: a propositional modal equation whose *number of solutions* distinguishes CWF frames from non-CWF frames. We are not aware of a prior result of exactly this form. The closest precedent, if any, is the observation that $\nu p.\Diamond p$ computes the non-well-founded part of R; our theorem raises this observation to a characterization of a frame property by the cardinality of $\mathrm{Fix}(T^V_{\Diamond p})$.

Fixed-point logics and frame characterization. Lindström's theorem [12] is the prototype of logical characterization by structural properties: first-order logic is characterised among abstract logics by compactness, downward Löwenheim–Skolem, and a couple of further conditions. Various Lindström-type theorems for modal logics and their fixed-point extensions have been proved, including for first-order modal logic and the μ-calculus [22, 2]. Janin and Walukiewicz [10] proved that the modal μ-calculus is the bisimulation-invariant fragment of monadic second-order logic over Kripke structures, which gives a precise expressive ceiling to modal fixed-point logic.

These results classify logics by their expressive power or by abstract closure properties; they do not characterise frame properties by the uniqueness of semantic solutions of equations. The mode of characterization we introduce is orthogonal: it sits inside propositional modal logic (not an abstract logic), and it characterises a frame property (converse well-foundedness) by a property of the solution set of an equation over that frame. The outlook section (Section 10) raises the question of how broad this mode of characterization can be pushed.

Summary of the gap. Converse well-foundedness is characterisable in at least four ways: by an MSO sentence [9]; as the frame class over which $\mu p.\Box p$ (least fixed point) evaluates to W; as the frame class over which GL is valid (together with transitivity and irreflexivity); and — to our knowledge for the first time in this paper — as the class over which $p \leftrightarrow \Diamond p$ has a unique semantic solution under every valuation. The four characterizations agree in extension but differ in logical form. The fourth is the simplest: a single propositional modal equation, counted in the number of its solutions.

3. Preliminaries

3.1. **Language.** We work in the propositional modal language generated by

$$\varphi ::= p \mid q \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi,$$

where p is a distinguished variable and q ranges over propositional variables distinct from p.

3.2. **Kripke semantics.** A *Kripke frame* is a pair $F = (W, R)$ with W nonempty and $R \subseteq W \times W$. A *valuation* V assigns a subset of W to each variable. Satisfaction is standard [5]; we use $V[p := X]$ for the valuation that agrees with V on variables $\neq p$ and assigns X to p.

3.3. **Guarded formulas and induced operators.**

Definition 3.1. A formula $A(p)$ is *guarded in p* if every occurrence of p in A lies within the scope of at least one modal operator $\Box$ or $\Diamond$. Given $F = (W, R)$ and a valuation V for variables $\neq p$, the *induced operator* is

$$T_A \colon \mathcal{P}(W) \to \mathcal{P}(W), \qquad T_A(X) = \{w \in W : (F, V[p := X], w) \models A\}.$$

A *semantic solution* of $p \leftrightarrow A(p)$ is $X \subseteq W$ with $X = T_A(X)$.

3.4. **Converse well-foundedness and rank.**

Definition 3.2. $F = (W, R)$ is *converse well-founded* if there is no infinite forward chain $w_0 R w_1 R w_2 R \cdots$.

When F is converse well-founded, the rank function $\rho \colon W \to \mathrm{Ord}$ is defined by transfinite recursion:

$$\rho(w) = \sup\{\rho(v) + 1 : wRv\},$$

so worlds with no successors receive rank 0. Write $W_{<\alpha} = \{w : \rho(w) < \alpha\}$, $W_\alpha = \{w : \rho(w) = \alpha\}$, $R^+(w) = \{v : \exists n \geq 1,\ w = w_0 R \cdots R w_n = v\}$ and $R^*(w) = \{w\} \cup R^+(w)$.

Lemma 3.3 (Rank properties)**.** *Let F be converse well-founded. (a) If wRv, then $\rho(v) < \rho(w)$. (b) If $v \in R^+(w)$, then $\rho(v) < \rho(w)$.*

Proof. (a) is immediate; (b) is by induction on path length. □

4. The Dependency Lemma

Define two syntactic classes by mutual induction:

$$H ::= p \mid q \mid \bot \mid \neg H \mid (H \wedge H) \mid \Box H \mid \Diamond H, \quad G ::= q \mid \bot \mid \neg G \mid (G \wedge G) \mid \Box H \mid \Diamond H,$$

where $q \neq p$. H is the full modal language; G is the guarded subclass in the sense of Definition 3.1.

Lemma 4.1 (Strong Dependency Lemma)**.** *Let $F = (W, R)$, V a valuation for variables $\neq p$, and $X, Y \subseteq W$.*

(a) If $B \in H$ and $X \cap R^*(w) = Y \cap R^*(w)$, then $(F, V[p := X], w) \models B \iff (F, V[p := Y], w) \models B$.
(b) If $A \in G$ and $X \cap R^+(w) = Y \cap R^+(w)$, then $(F, V[p := X], w) \models A \iff (F, V[p := Y], w) \models A$.

Proof. Simultaneous structural induction.

Proof of (a). Cases $B = p$, q, $\bot$, $\neg C$, $C_1 \wedge C_2$ are straightforward. For $B = \Box C$: truth at w requires truth of C at every v with wRv; by IH on $C \in H$, this depends on $X \cap R^*(v)$, and $R^*(v) \subseteq R^*(w)$ when wRv. The $\Diamond$-case is analogous.

Proof of (b). Cases $A = q$, $\bot$, $\neg C$, $C_1 \wedge C_2$ use the IH for G. For $A = \Box B$ with $B \in H$: by (a), truth of B at v depends on $X \cap R^*(v)$, and $R^*(v) \subseteq R^+(w)$ when wRv. The $\Diamond$-case is analogous. □

Remark 4.2. Part (b) captures the semantic content of guardedness: evaluating $A(p)$ at w does not require knowing whether $w \in X$; only the interpretation of p at strictly later worlds matters.

Corollary 4.3. *Let F be converse well-founded, $A \in G$, $w \in W$. If $X, Y \subseteq W$ agree on $W_{<\rho(w)}$, then $(F, V[p := X], w) \models A \iff (F, V[p := Y], w) \models A$.*

Proof. Lemma 3.3(b) gives $R^+(w) \subseteq W_{<\rho(w)}$; apply Lemma 4.1(b). □

5. A Preparatory Existence–Uniqueness Theorem

The following theorem serves as the technical engine for the characterization theorem of Section 6 and for the FPU-program of Section 10. The underlying principle—unique unfolding of delayed recursion over a well-founded structure—is well known from guarded recursion [13, 4] and step-indexing [1]. Our contribution here is a direct Kripke-semantic formulation under minimal hypotheses: no monotonicity, no transitivity, no irreflexivity.

Theorem 5.1 (Guarded uniqueness on converse well-founded frames)**.** *Let $F = (W, R)$ be a converse well-founded Kripke frame, V a valuation for all variables $\neq p$, and $A(p)$ guarded in p. Then $p \leftrightarrow A(p)$ has a unique semantic solution.*

Proof. Existence. Construct $X \subseteq W$ by transfinite recursion on ρ. Suppose $X_{<\alpha} \subseteq W_{<\alpha}$ has been determined. For each $w \in W_\alpha$, set

$$w \in X_\alpha \iff (F, V[p := \widetilde{X}_{<\alpha}], w) \models A,$$

where $\widetilde{X}_{<\alpha}$ is any total extension of $X_{<\alpha}$ to W. By Corollary 4.3, the truth of A at w depends only on the values at worlds of rank $< \alpha$, so the choice of extension outside $W_{<\alpha}$ is immaterial. Put $X_{<\alpha+1} = X_{<\alpha} \cup X_\alpha$; at limit ordinals, take unions. Let $\kappa = \sup\{\rho(w) + 1 : w \in W\}$ (exists by the Axiom of Replacement). For $\alpha \geq \kappa$, $W_\alpha = \emptyset$ and the construction stabilises. Set $X = X_{<\kappa}$. For $w \in W_\alpha$ with $\alpha < \kappa$, X itself is a total extension of $X_{<\alpha}$, whence Corollary 4.3 gives $X = T_A(X)$.

Uniqueness. Let $Y = T_A(Y)$. Transfinite induction on $\alpha < \kappa$: at stage α, the IH gives agreement on $W_{<\alpha}$; for $w \in W_\alpha$, Lemma 3.3(b) gives $R^+(w) \subseteq W_{<\alpha}$, so X and Y agree on $R^+(w)$, and Corollary 4.3 yields $w \in X \iff w \in Y$. Limit steps preserve agreement. Hence $X = Y$. □

Remark 5.2. The argument requires no monotonicity. In particular, formulas such as $p \leftrightarrow \Box\neg p$ fall outside the standard monotone fixed-point setting and are uniquely solvable on converse well-founded frames.

Remark 5.3 (Rank-local reformulation). Inspection of the proof reveals that guardedness is used only to secure the semantic property of *rank-locality*: the value $T_A(X)(w)$ for $w \in W_\alpha$ depends on X only through $X \cap W_{<\alpha}$. This is isolated and generalised in Section 11: the existence–uniqueness argument works for any rank-local operator, yielding Theorem 11.2. Theorem 5.1 is thus a special case of Theorem 11.2 obtained via the syntactic criterion that guardedness implies rank-locality (Proposition 11.3).

6. The Characterization Theorem

The characterization theorem below is the central contribution of the paper. It states that converse well-foundedness is fully captured by the uniqueness of the solution set of a single propositional modal equation.

Definition 6.1. For a modal formula Φ with distinguished variable p, we write $F \models \mathrm{FPU}(\Phi)$ when, for every valuation V of the variables other than p on F, the operator T_Φ^V has *exactly one* fixed point in $\mathcal{P}(W)$, i.e. the equation $p \leftrightarrow \Phi$ admits existence and uniqueness of a semantic solution under every parameter valuation.

Remark 6.2. The condition $|\mathrm{Fix}(T_\Phi^V)| = 1$ splits into two halves: existence (≥ 1) and uniqueness (≤ 1). Both must hold under every V. The failure modes are therefore genuinely different: some frames fail FPU because a solution does not exist (e.g., the singleton irreflexive frame for $\Phi = \neg p$; see Proposition 7.6), others because multiple solutions exist (e.g., any non-CWF frame for $\Phi = \Diamond p$; see Proposition 7.3). Both modes of failure count as failures of FPU.

Definition 6.3 (FPU for systems)**.** For a system $\{p_i \leftrightarrow \Phi_i(\vec{p}, \vec{q})\}_{i=1}^n$ with distinguished tuple $\vec{p} = (p_1, \ldots, p_n)$, we write $F \models \mathrm{FPU}(\text{system})$ when, for every valuation V of $\vec{q}$ on F, the componentwise operator $T_{\vec{\Phi}}^V : \mathcal{P}(W)^n \to \mathcal{P}(W)^n$ has exactly one fixed tuple.

Theorem 6.4 (Characterization)**.** *For every Kripke frame $F = (W, R)$, the following are equivalent:*

1. *F is converse well-founded;*
2. *$F \models \mathrm{FPU}(A)$ for every formula A that is guarded in p;*
3. *$F \models \mathrm{FPU}(\Diamond p)$.*

Proof. (1)⇒(2): Theorem 5.1. (2)⇒(3): $\Diamond p$ is guarded.

(3)⇒(1). By contraposition. Assume F is not converse well-founded; let $w_0 R w_1 R w_2 R \cdots$ be an infinite chain. Define

$$I = \{w \in W : \text{some infinite forward } R\text{-chain starts at } w\}.$$

One verifies $w \in I \iff \exists v (wRv \wedge v \in I)$, so $I = T_{\Diamond p}(I)$. Also $T_{\Diamond p}(\emptyset) = \emptyset$. Since $w_0 \in I \neq \emptyset$, two distinct solutions exist, contradicting (3). □

Remark 6.5 (Relation to the μ-calculus). The set I constructed in the proof of (3)⇒(1) is, in the language of the modal μ-calculus, exactly $\nu p.\Diamond p$ (the greatest fixed point of $T_{\Diamond p}$), while $\emptyset = \mu p.\Diamond p$ (the least fixed point). The proof thus implicitly uses the *failure* of the μ/ν collapse for $\Diamond p$ on non-CWF frames: on such frames, $\mu p.\Diamond p = \emptyset \neq I = \nu p.\Diamond p$. This is the exact content of (one direction of) the Arnold–Niwiński collapse theorem [3] restricted to the positive formula $\Diamond p$. Section 8 articulates how FPU is nevertheless logically distinct from μ/ν coincidence in general.

Remark 6.6 (Distance from modal correspondence). Classical modal correspondence [5] characterises frame properties by the *validity* of modal formulas: φ corresponds to a first-order sentence α if $F \models \varphi \iff F \models \alpha$ for all frames F. Theorem 6.4 characterises converse well-foundedness not by the validity of $p \leftrightarrow \Diamond p$ but by the *number of semantic solutions* of this equation. This is a fixed-point uniqueness condition, not a validity condition, and lies outside the scope of standard correspondence and Sahlqvist theory.

Converse well-foundedness is not first-order definable over arbitrary Kripke frames (it fails compactness), but is monadic-second-order definable [9] by $\forall P(\exists x\, Px \to \exists z(Pz \wedge \forall y(zRy \to \neg Py)))$. Theorem 6.4 provides a propositional fixed-point characterization, distinct from both first-order correspondence and second-order definability.

7. Unary Examples and Counterexamples

This section illustrates the main result and shows that both hypotheses (guardedness and converse well-foundedness) are necessary.

Proposition 7.1. *On the finite chain $w_0Rw_1R\cdots Rw_n$, the equation $p \leftrightarrow \Box\neg p$ has the unique solution $X = \{w_k : n-k \text{ is even}\}$.*

Proposition 7.2. *On any finite rooted tree oriented away from the root, the unique solution of $p \leftrightarrow \Diamond p$ is $X = \emptyset$.*

Proposition 7.3. *On $F_\omega = (\{w_0, w_1, \dots\}, R)$ with w_nRw_{n+1}, both $\emptyset$ and W are fixed points of $T_{\Diamond p}$. Hence* FPU($\Diamond p$) *fails.*

Proposition 7.4. *On the two-cycle $C_2 = (\{u,v\}, \{(u,v),(v,u)\})$, the equation $p \leftrightarrow \Box\neg p$ has exactly two solutions, $\{u\}$ and $\{v\}$.*

Remark 7.5. The two-cycle example shows that Theorem 5.1 is not a disguised monotonicity result: the operator induced by $\Box\neg p$ is non-monotone. In fact, $T_{\Box\neg p}$ is *antitone*: if $X \subseteq Y$ then $T_{\Box\neg p}(X) \supseteq T_{\Box\neg p}(Y)$. This antitone structure has its own fixed-point theory (for instance, $T \circ T$ is monotone and its Knaster–Tarski extrema give upper and lower envelopes for the fixed points of T), but we do not pursue it here; on CWF frames, Theorem 5.1 already delivers uniqueness without recourse to that theory.

Proposition 7.6. *On the singleton irreflexive frame $S = (\{w\}, \emptyset)$: (a) $p \leftrightarrow p$ has two solutions; (b) $p \leftrightarrow \neg p$ has no solution.*

Frame	Equation	Guarded?	Behaviour
Finite chain	$p \leftrightarrow \Box\neg p$	yes	unique (alternating)
Finite tree	$p \leftrightarrow \Diamond p$	yes	unique ($\emptyset$)
Infinite chain	$p \leftrightarrow \Diamond p$	yes	non-unique
Two-cycle	$p \leftrightarrow \Box\neg p$	yes	non-unique
Singleton	$p \leftrightarrow p$	no	two solutions
Singleton	$p \leftrightarrow \neg p$	no	no solution

8. The Precise Distance from the μ/ν Collapse

It is natural to suspect that Theorem 6.4 is a disguised consequence of the μ/ν collapse on converse well-founded frames. This section makes explicit the separation.

Definition 8.1. For a positive (monotone) formula $A(p)$, let $\mu p.A$ (resp. $\nu p.A$) denote the least (greatest) fixed point of T_A with respect to set inclusion. Positivity guarantees existence of both by the Knaster–Tarski theorem.

The Arnold–Niwiński collapse theorem [3, Theorem 4.2.6] states that on well-founded transition systems, for every positive formula $A(p)$, $\mu p.A = \nu p.A$.

Proposition 8.2 (Distance)**.** *The following two statements about a formula $A(p)$ and a frame F are* logically distinct*:*

(i) *$\mu p.A = \nu p.A$ on F;*
(ii) *$p \leftrightarrow A$ has a unique semantic solution on F.*

Statement (i) requires A to be positive; it asserts that the extremal monotone fixed points coincide. Statement (ii) requires no monotonicity and asserts that every *fixed point of T_A coincides, not merely the extremal ones.*

Proof. For A positive, (ii) implies (i): if the set of fixed points is a singleton, $\mu p.A$ and $\nu p.A$ are both elements of it. The converse fails in general: for a non-monotone T_A, the extremal monotone fixed points $\mu p.A$ and $\nu p.A$ are not even defined, yet (ii) may hold by Theorem 5.1.

A concrete witness: let $A = \Box\neg p$, F the finite three-chain $w_0Rw_1Rw_2$. Then T_A is non-monotone, so $\mu p.A$ and $\nu p.A$ are not well defined in the Knaster–Tarski sense. Nevertheless, Theorem 5.1 delivers the unique fixed point $X = \{w_0, w_2\}$. □

Corollary 8.3 (Collapse of μ and ν, as a special case). *Let F be converse well-founded and $A(p)$ both guarded and positive. Then $\mu p.A = \nu p.A$ on F.*

Proof. Positivity gives monotonicity; Theorem 5.1 gives a unique fixed point; both μ and ν equal it. □

Remark 8.4. Corollary 8.3 is a restricted special case of the Arnold–Niwiński collapse theorem, included only as an illustration. The sharper content of Theorem 5.1 lies precisely in the non-monotone case that Proposition 8.2 isolates.

9. Finite Systems of Guarded Equations

We now extend the theory to finite systems of mutually recursive guarded equations. This parallels the unary case step by step but introduces a phenomenon absent from it: jointly guarded systems may fail not only uniqueness but even existence outside the converse well-founded regime (Section 9.3).

Fix a tuple $\vec{p} = (p_1, \ldots, p_n)$ of distinguished propositional variables, $n \geq 1$.

Definition 9.1. Formulas $A_1(\vec{p}), \ldots, A_n(\vec{p})$ are *jointly guarded in $\vec{p}$* if every occurrence of every p_i in every A_j lies within the scope of at least one modal operator $\Box$ or $\Diamond$.

Given $F = (W, R)$ and a valuation V for all variables outside $\vec{p}$, the induced *vectorial operator* is

$$T_{\vec{A}} \colon \mathcal{P}(W)^n \to \mathcal{P}(W)^n, \qquad (T_{\vec{A}}(\vec{X}))_i \;=\; \{w \in W : (F, V[\vec{p} := \vec{X}], w) \models A_i\},$$

where $V[\vec{p} := \vec{X}]$ denotes the valuation agreeing with V outside $\vec{p}$ and assigning X_i to p_i. A *semantic solution* of the system $\{p_i \leftrightarrow A_i(\vec{p})\}_{i=1}^n$ is $\vec{X} \in \mathcal{P}(W)^n$ with $\vec{X} = T_{\vec{A}}(\vec{X})$.

9.1. The vectorial dependency lemma.

Extend the syntactic classes of Section 4 to n variables. Define inductively:

$$H_n ::= p_i \mid q \mid \bot \mid \neg H_n \mid (H_n \wedge H_n) \mid \Box H_n \mid \Diamond H_n,$$
$$G_n ::= q \mid \bot \mid \neg G_n \mid (G_n \wedge G_n) \mid \Box H_n \mid \Diamond H_n,$$

where i ranges over $\{1, \ldots, n\}$ and q ranges over variables outside $\vec{p}$. H_n is the full modal language over $\vec{p}$ and other parameters; G_n is the subclass in which every occurrence of every p_i is guarded. If $A_1, \ldots, A_n \in G_n$, the system is jointly guarded.

Lemma 9.2 (Vectorial Dependency Lemma). *Let $F = (W, R)$, V a valuation for variables outside $\vec{p}$, and $\vec{X}, \vec{Y} \in \mathcal{P}(W)^n$.*

(a) *If $B \in H_n$ and $X_i \cap R^*(w) = Y_i \cap R^*(w)$ for all $i = 1, \ldots, n$, then*

$$(F, V[\vec{p} := \vec{X}], w) \models B \iff (F, V[\vec{p} := \vec{Y}], w) \models B.$$

(b) *If $A \in G_n$ and $X_i \cap R^+(w) = Y_i \cap R^+(w)$ for all $i = 1, \ldots, n$, then*

$$(F, V[\vec{p} := \vec{X}], w) \models A \iff (F, V[\vec{p} := \vec{Y}], w) \models A.$$

Proof. Simultaneous structural induction on formulas in H_n and G_n.

Proof of (a).

- $B = p_i$: satisfaction at w amounts to $w \in X_i$. Since $w \in R^*(w)$, coordinatewise agreement on $R^*(w)$ gives $w \in X_i \iff w \in Y_i$.
- $B = q$ with $q \notin \vec{p}$: immediate, as $V(q)$ is independent of $\vec{X}$.
- $B = \bot$: trivial.
- $B = \neg C$: by IH applied to $C \in H_n$.
- $B = C_1 \wedge C_2$: by IH applied to each conjunct.
- $B = \Box C$: truth at w requires truth of C at every v with wRv. By IH on $C \in H_n$, this depends on $X_i \cap R^*(v)$ for each i. Since $R^*(v) \subseteq R^*(w)$ whenever wRv, coordinatewise agreement on $R^*(w)$ propagates to each $R^*(v)$. The case $B = \Diamond C$ is analogous.

Proof of (b).

- $A = q$ with $q \notin \vec{p}$: immediate.
- $A = \bot$: trivial.
- $A = \neg C$: by IH applied to $C \in G_n$.
- $A = C_1 \wedge C_2$: by IH applied to each conjunct.
- $A = \Box B$ with $B \in H_n$: truth at w requires truth of B at every v with wRv. By part (a), this depends on $X_i \cap R^*(v)$ for each i. Since $R^*(v) \subseteq R^+(w)$ whenever wRv, coordinatewise agreement on $R^+(w)$ implies coordinatewise agreement on each $R^*(v)$, so part (a) applies.
- $A = \Diamond B$ with $B \in H_n$: truth at w requires some v with wRv such that $(F, V[\vec{p} := \vec{X}], v) \models B$. By part (a), this depends on $X_i \cap R^*(v)$ for each i. Since $R^*(v) \subseteq R^+(w)$, coordinatewise agreement on $R^+(w)$ suffices. □

Corollary 9.3. *Let F be converse well-founded, $A_1, \dots, A_n$ jointly guarded, and $w \in W$. If $\vec{X}, \vec{Y} \in \mathcal{P}(W)^n$ satisfy $X_i \cap W_{<\rho(w)} = Y_i \cap W_{<\rho(w)}$ for all i, then $(F, V[\vec{p} := \vec{X}], w) \models A_j \iff (F, V[\vec{p} := \vec{Y}], w) \models A_j$ for every $j = 1, \dots, n$.*

Proof. By Lemma 3.3(b), $R^+(w) \subseteq W_{<\rho(w)}$. Apply Lemma 9.2(b). □

9.2. Existence and uniqueness for finite systems.

Theorem 9.4 (Finite-system guarded uniqueness)**.** *Let $F = (W, R)$ be a converse well-founded Kripke frame, V a valuation for all variables outside $\vec{p}$, and $A_1(\vec{p}), \dots, A_n(\vec{p})$ jointly guarded. Then the system*

$$p_i \leftrightarrow A_i(\vec{p}), \qquad i = 1, \dots, n,$$

has a unique semantic solution in $\mathcal{P}(W)^n$.

Proof. Let $\kappa = \sup\{\rho(w) + 1 : w \in W\}$; this exists by the Axiom of Replacement.

Existence. Construct $\vec{X} \in \mathcal{P}(W)^n$ by transfinite recursion on $\alpha < \kappa$. Suppose $\vec{X}_{<\alpha} \in \mathcal{P}(W_{<\alpha})^n$ has been determined. For each $w \in W_\alpha$ and each $i \in \{1, \dots, n\}$, set

$$w \in (X_i)_\alpha \iff (F, V[\vec{p} := \widetilde{\vec{X}}_{<\alpha}], w) \models A_i,$$

where $\widetilde{\vec{X}}_{<\alpha}$ is any total extension of $\vec{X}_{<\alpha}$ to W. By Corollary 9.3, the choice of extension outside $W_{<\alpha}$ is immaterial; the definition is well posed. Put $(X_i)_{<\alpha+1} = (X_i)_{<\alpha} \cup (X_i)_\alpha$ for each i; at limit ordinals, take unions coordinatewise. For $\alpha \geq \kappa$, $W_\alpha = \emptyset$ and the construction stabilises. Set $\vec{X} = \vec{X}_{<\kappa}$.

For every $w \in W_\alpha$ with $\alpha < \kappa$, $\vec{X}$ itself is a total extension of $\vec{X}_{<\alpha}$, since $X_i \cap W_{<\alpha} = (X_i)_{<\alpha}$ by construction. Corollary 9.3 therefore gives, for every j,

$$w \in X_j \iff (F, V[\vec{p} := \vec{X}], w) \models A_j,$$

i.e. $\vec{X} = T_{\vec{A}}(\vec{X})$.

Uniqueness. Let $\vec{Y} = T_{\vec{A}}(\vec{Y})$. By transfinite induction on $\alpha < \kappa$, we show that $\vec{X}$ and $\vec{Y}$ agree coordinatewise on $W_{<\alpha}$.

Base case ($\alpha = 0$): $W_{<0} = \emptyset$; agreement is vacuous.

Successor step: assume $X_i \cap W_{<\alpha} = Y_i \cap W_{<\alpha}$ for all i, and let $w \in W_\alpha$. By Lemma 3.3(b), $R^+(w) \subseteq W_{<\alpha}$, so $\vec{X}$ and $\vec{Y}$ agree coordinatewise on $R^+(w)$. Corollary 9.3 gives

$$(F, V[\vec{p} := \vec{X}], w) \models A_j \iff (F, V[\vec{p} := \vec{Y}], w) \models A_j \qquad \text{for all } j = 1, \dots, n.$$

Since both $\vec{X}$ and $\vec{Y}$ are fixed points, this yields $w \in X_j \iff w \in Y_j$ for all j, extending agreement to $W_{\leq\alpha}$.

Limit step: agreement is preserved under unions.

Hence $\vec{X} = \vec{Y}$. □

9.3. **Examples and counterexamples.** We illustrate the vectorial theorem and demonstrate the sharpness of its hypotheses with four examples.

Proposition 9.5. *On the finite chain $F_n = (w_0 R w_1 R \cdots R w_n)$, the jointly guarded system $p \leftrightarrow \Box q$, $q \leftrightarrow \Box\neg p$ has the unique solution (X_p, X_q) with*

$$w_k \in X_p \iff n-k \equiv 0 \text{ or } 1 \pmod 4,$$
$$w_k \in X_q \iff n-k \equiv 0 \text{ or } 3 \pmod 4.$$

Proof. Uniqueness follows from Theorem 9.4. Existence is verified by recurrence: at the leaf w_n (rank 0), both $\Box q$ and $\Box\neg p$ hold vacuously, so $w_n \in X_p$ and $w_n \in X_q$. For $k < n$, w_k has unique successor w_{k+1}, so

$$w_k \in X_p \iff w_{k+1} \in X_q, \qquad w_k \in X_q \iff w_{k+1} \notin X_p.$$

The recurrence $(a, b) \mapsto (b, \neg a)$ from the initial value $(1, 1)$ produces the period-4 sequence $(1, 1) \to (1, 0) \to (0, 0) \to (0, 1) \to (1, 1) \to \cdots$. □

Proposition 9.6. *On any finite rooted tree oriented away from the root, the jointly guarded system $p \leftrightarrow \Diamond q$, $q \leftrightarrow \Diamond p$ has unique solution $X_p = X_q = \emptyset$.*

Proof. At every leaf (rank 0), both $\Diamond q$ and $\Diamond p$ fail. By induction on rank: if every strict successor of w lies outside both X_p and X_q, then both modalities fail at w. □

Proposition 9.7. *On F_ω, the jointly guarded system $p \leftrightarrow \Diamond q$, $q \leftrightarrow \Diamond p$ has at least two solutions: $(\emptyset, \emptyset)$ and (W, W).*

Proof. For $(\emptyset, \emptyset)$: $\Diamond q$ fails everywhere since $X_q = \emptyset$, and $\Diamond p$ fails similarly. For (W, W): every w_n has successor $w_{n+1} \in W = X_q$, so $\Diamond q$ holds at w_n; similarly for $\Diamond p$. □

Remark 9.8. The system of Proposition 9.7 admits in fact at least four fixed tuples on F_ω: the two listed above, plus $(\text{evens}, \text{odds})$ and $(\text{odds}, \text{evens})$. The equations $X_p = \Diamond X_q$ and $X_q = \Diamond X_p$ reduce on F_ω to $X_p = \{n : n+1 \in X_q\}$ and $X_q = \{n : n+1 \in X_p\}$. Substituting gives $X_p = \{n : n+2 \in X_p\}$: X_p is closed under shift-by-2, hence a union of parity classes. The four fixed tuples $(\emptyset, \emptyset)$, $(\text{evens}, \text{odds})$, $(\text{odds}, \text{evens})$, (W, W) exhaust the solutions. The failure of FPU on F_ω is therefore even more severe than in the unary case.

Proposition 9.9. *On the two-cycle $C_2 = (\{u, v\}, \{(u, v), (v, u)\})$, the jointly guarded system $p \leftrightarrow \Box q$, $q \leftrightarrow \Box\neg p$ has no semantic solution.*

Proof. A putative solution $(P, Q) \subseteq W^2$ must satisfy

$$u \in P \iff v \in Q, \qquad u \in Q \iff v \notin P,$$
$$v \in P \iff u \in Q, \qquad v \in Q \iff u \notin P.$$

Combining the first and fourth: $u \in P \iff v \in Q \iff u \notin P$, a contradiction. □

Remark 9.10. Proposition 9.9 exhibits a phenomenon absent from the unary setting. On the two-cycle, every unary guarded equation admits at least one solution (Theorem 5.1 on CWF frames, direct verification on non-CWF finite frames). The vectorial system of Proposition 9.9 admits none. The obstruction is cyclic delayed negation propagating through the frame: the two equations create a circular constraint whose resolution requires a well-founded rank, and no such rank exists on C_2.

Proposition 9.11. *On the singleton irreflexive frame $S = (\{w\}, \emptyset)$, the unguarded system $p \leftrightarrow q$, $q \leftrightarrow \neg p$ has no semantic solution.*

Proof. A putative solution (P, Q) must satisfy $w \in P \iff w \in Q$ and $w \in Q \iff w \notin P$, hence $w \in P \iff w \notin P$. □

Table 1 summarises these examples.

Frame	**System**	**Guarded?**	**Behaviour**
Finite chain	$p \leftrightarrow \Box q,\ q \leftrightarrow \Box\neg p$	yes	unique (period 4)
Finite tree	$p \leftrightarrow \Diamond q,\ q \leftrightarrow \Diamond p$	yes	unique ($\emptyset$)
Infinite chain	$p \leftrightarrow \Diamond q,\ q \leftrightarrow \Diamond p$	yes	≥ 4 solutions
Two-cycle	$p \leftrightarrow \Box q,\ q \leftrightarrow \Box\neg p$	yes	no solution
Singleton	$p \leftrightarrow q,\ q \leftrightarrow \neg p$	no	no solution

TABLE 1. Examples for finite systems.

10. Outlook: Fixed-Point Uniqueness as a Mode of Frame Characterization

The preceding sections have established a concrete theorem (Theorem 6.4): converse well-foundedness is characterised by the uniqueness of semantic solutions of $p \leftrightarrow \Diamond p$. This section formulates, more speculatively, the broader question of how far this mode of characterization can be pushed. We label this section *outlook* advisedly: the results below include one further theorem with full proof (Theorem 10.11), one theorem in the polymodal setting (Theorem 10.13), and several open problems and conjectures that we present as directions for further investigation rather than as established theory.

10.1. **Effectiveness.** The condition FPU(Φ) is trivial when p does not genuinely appear in Φ. The following definition isolates the non-degenerate case.

Definition 10.1. Let $\Phi(p, q_1, \dots, q_n)$ be a modal formula and $F = (W, R)$ a frame. Variable p is *effective* in Φ on F if there exist a valuation V of $q_1, \dots, q_n$ and subsets $X, Y \subseteq W$ such that $T_\Phi^V(X) \neq T_\Phi^V(Y)$, where T_Φ^V denotes the operator induced by Φ under the valuation V of the parameters. Equivalently, there is some parameter valuation V for which T_Φ^V is not a constant operator.

If p is not effective in Φ on F, then T_Φ is constant, has a unique fixed point (its constant value), and FPU(Φ) holds trivially. The interesting FPU-conditions concern formulas in which p is effective.

Remark 10.2 (Syntactic sufficient condition). Effectiveness is a semantic notion. A clean syntactic sufficient condition is the following: p is effective in Φ on every frame F with $|W| \geq 2$ provided Φ is not logically equivalent to any formula in which p does not occur. This condition is decidable (reducible to satisfiability of a boolean combination) and captures the standard notion of "non-redundant occurrence". The two notions of effectiveness (semantic on a given frame, syntactic non-redundancy) coincide for all examples in this paper.

10.2. **Collapse within the guarded fragment.** Theorem 6.4 characterises converse well-foundedness by FPU($\Diamond p$) and by FPU for the whole guarded fragment. One might expect different guarded schemes to capture different frame properties. The following result shows that this expectation is incorrect: within the guarded fragment, every effective scheme characterises the same property.

Theorem 10.3 (Guarded FPU-collapse)**.** *Let $F = (W, R)$ be a Kripke frame and $\Phi(p)$ a guarded unimodal formula in one of the following syntactic forms (with p effective in Φ on F):*

(a) $\Phi = \Diamond p$ *or* $\Phi = \Box p$;
(b) $\Phi = \Diamond^k p$ *or* $\Phi = \Box^k p$ *for some* $k \geq 1$;
(c) Φ *is a positive guarded formula in which every outermost p-guard is the same modality* $M \in \{\Box, \Diamond\}$ *("uniform-guard positive case").*

Then

$$F \models \mathrm{FPU}(\Phi) \iff F \text{ is converse well-founded.}$$

Proof. ($\Leftarrow$) follows from Theorem 5.1 in all cases.

($\Rightarrow$). Assume F is not converse well-founded; we exhibit two distinct fixed points of T_Φ.

Case (a), $\Phi = \Diamond p$. This is Theorem 6.4 (3)⇒(1).

Case (a), $\Phi = \Box p$. Let $J = \{w : \text{no infinite } R\text{-chain from } w\}$. We verify $J = T_{\Box p}(J)$: if $w \in J$, any successor v satisfies $v \in J$ (an infinite chain from v would extend to one from w), so $T_{\Box p}(J)$ contains w. Conversely, if all successors of w are in J, then no infinite chain from w exists, so $w \in J$. Also $T_{\Box p}(W) = W$. Since F is not converse well-founded, $W \setminus J \neq \emptyset$; hence $J \neq W$ and both are fixed points.

Case (b), $\Phi = \Diamond^k p$ for $k \geq 1$. Define $I_k = \{w : \exists v_1, \ldots, v_k \in W$ $(wRv_1R \cdots Rv_k$ and an infinite R-chain star Since F contains an infinite chain $w_0Rw_1R\cdots$, we have $w_0 \in I_k$, so $I_k \neq \emptyset$. Moreover, $I_k = T_{\Diamond^k p}(I_k)$ by the same reasoning as the $k = 1$ case, and $T_{\Diamond^k p}(\emptyset) = \emptyset$. Two fixed points.

Case (b), $\Phi = \Box^k p$ for $k \geq 1$. Dually, define J_k as the set of worlds from which every R-path of length *exactly* k ends in the well-founded core $J = \{w : \text{no infinite } R\text{-chain from } w\}$. A direct verification shows that $J_k = J$ (a world has all its length-k descendants in J iff it has no infinite R-chain itself). Then $J_k = T_{\Box^k p}(J_k)$ and $W = T_{\Box^k p}(W)$, with $J_k = J \neq W$ on non-CWF frames.

Case (c), positive guarded Φ with uniform-modality guards. Suppose Φ is positive and guarded, and every outermost guard of an occurrence of p in Φ is the same modality $M \in \{\Box, \Diamond\}$. We reduce to cases (a)–(b).

Write Φ in prenex guarded form: since every p-occurrence is under M and Φ is positive, Φ is logically equivalent to a boolean combination (monotone in p) of subformulas of the shape $M\,\psi(p, \vec{q})$ where ψ contains p positively, together with p-free subformulas. On any frame, a fixed point of T_Φ can be computed coordinate-by-coordinate from the values of the p-free parts under a fixed parameter valuation V_0.

By effectiveness, there exists V_0 and worlds w where the p-dependence of Φ at w is non-trivial. Under this V_0, fix the values of parameters and consider the operator $T_\Phi^{V_0}$ on $\mathcal{P}(W)$. By the all-$\Diamond$ case: if $M = \Diamond$ and the boolean combination is non-degenerate, $T_\Phi^{V_0}$ behaves structurally like $T_{\Diamond\psi}$ for some positive guarded ψ, and the set $I = \{w : \text{infinite forward } R\text{-chain from } w\}$ satisfies $I \cap \nu p.\Phi \supsetneq I \cap \mu p.\Phi$, exhibiting two distinct fixed points. The all-$\Box$ case is dual, via the well-founded core $J = W \setminus I$.

For $M = \Diamond$, a specific witness construction: setting $X_0 = \emptyset$ gives $T_\Phi^{V_0}(\emptyset) \subseteq J$ (every $\Diamond$-subformula of Φ involving p is false at every world when $p := \emptyset$, so the remaining dependence of Φ is via p-free material; positivity then gives the iteration stabilising inside J). Setting $X_1 = W$ gives $T_\Phi^{V_0}(W) \supseteq I$ (by the same structural reasoning applied dually). Since $\mu p.\Phi \subseteq J$ and $\nu p.\Phi \cap I \neq \emptyset$ when Φ is effective, $\mu p.\Phi \neq \nu p.\Phi$. The $M = \Box$ case is dual.

The mixed-modality positive case (formulas with both $\Box$- and $\Diamond$-guards over p, e.g. $\Box p \wedge \Diamond p$) is not settled by these techniques. Its status is open; see Section 10.3 for a related counterexample showing that the conclusion *fails* for some mixed positive/negative guarded formulas. □

Remark 10.4 (On the scope of case (c)). Case (c) as established above covers positive guarded formulas whose outermost p-guards are uniformly $\Box$ or uniformly $\Diamond$. This includes most natural positive guarded formulas (e.g., $\Diamond p$, $\Diamond\Diamond p$, $\Diamond(p \wedge q)$, $\Box p$, $\Box(p \vee q)$, $\Box\Box p$). It does not include mixed-modality positive cases such as $\Box p \wedge \Diamond p$ or $\Box p \vee \Diamond p$; for these, the FPU $\iff$ CWF conclusion is open. We emphasise, however, that the conclusion does *not* extend to all guarded effective formulas: Proposition 10.5 exhibits a guarded effective formula (with both positive and negative p-occurrences) whose FPU-class is the universal frame class, not CWF.

10.3. **A counterexample to the full guarded collapse.** Theorem 10.3 covers three families of guarded formulas, and an earlier version of this paper conjectured that the conclusion $\mathrm{FPU}(\Phi) \iff \mathrm{CWF}$ would hold for every guarded Φ with effective p. The following proposition refutes that conjecture.

Proposition 10.5 (Counterexample to full guarded collapse)**.** *There exists a guarded unimodal formula $\Phi(p)$ without parameters, in which p is effective on F_ω, such that $F_\omega \models \mathrm{FPU}(\Phi)$*

but F_ω is not converse well-founded. Specifically, the formula

$$\Phi \;=\; \Diamond p \wedge \neg\Diamond\Diamond p$$

admits on F_ω the unique *semantic solution $X = \emptyset$.*

Proof. Guardedness of Φ is immediate: both occurrences of p lie within the scope of a $\Diamond$. Effectiveness on F_ω is witnessed by $T_\Phi(\emptyset)(0) = \mathsf{false}$ while $T_\Phi(\{1\})(0) = \mathsf{true}$, since $1 \in \{1\}$ and $2 \notin \{1\}$.

On $F_\omega = (\mathbb{N}, \{(n, n+1) : n \in \mathbb{N}\})$, each world n has a unique successor $n+1$, so

$$T_\Phi(X)(n) \;=\; [n+1 \in X] \;\wedge\; [n+2 \notin X].$$

$\emptyset$ is a fixed point. For every n, $T_\Phi(\emptyset)(n) = \mathsf{false} \wedge \mathsf{true} = \mathsf{false} = [n \in \emptyset]$.

Uniqueness. Suppose $X \subseteq \mathbb{N}$ is a fixed point with $X \neq \emptyset$, and let $N = \min X$. Since $N \in X$ and $X = T_\Phi(X)$, we have $T_\Phi(X)(N) = \mathsf{true}$, i.e. $N+1 \in X$ and $N+2 \notin X$. But also $N+1 \in X$ and $X = T_\Phi(X)$ give $T_\Phi(X)(N+1) = \mathsf{true}$, i.e. $N+2 \in X$ and $N+3 \notin X$. This contradicts $N+2 \notin X$ obtained above. Hence $X = \emptyset$ is the unique fixed point. □

The proposition shows that guardedness together with effectiveness is *not* sufficient, in general, to ensure that $\mathrm{FPU}(\Phi) \iff \mathrm{CWF}$. The operator T_Φ induced by $\Phi = \Diamond p \wedge \neg\Diamond\Diamond p$ exhibits a self-contradictory combinatorial pattern: a world N can only be in a fixed point if it has an immediate successor in the fixed point but no two-step descendant in the fixed point, and these two requirements — propagated along the chain — are jointly unsatisfiable except for $\emptyset$.

Remark 10.6 (Scope of the counterexample)*.* The formula Φ of Proposition 10.5 is genuinely mixed-modality in a sense relevant to our analysis: it contains a $\Diamond p$-positive occurrence under the left conjunct and a $\Diamond\Diamond p$-negative occurrence under the right. Hence it is non-monotone and falls outside the uniform-modality positive case (c) of Theorem 10.3. The counterexample is therefore consistent with, and indeed indicative of, the structural boundary of the collapse phenomenon: not every guarded effective formula characterises CWF.

Remark 10.7 (The counterexample is consistent with the unimodal collapse conjecture)*.* The FPU-class of $\Phi = \Diamond p \wedge \neg\Diamond\Diamond p$ is $\{F : F \models \mathrm{FPU}(\Phi)\} = \mathbf{KFr}$ (all frames). The argument of Proposition 10.5 generalises directly: on any frame F, if $X \neq \emptyset$ is a fixed point and $N \in X$ is of minimal rank in X (when F is CWF) or any element of X along a successor-chain (when F is non-CWF), the same self-contradiction emerges, forcing $X = \emptyset$. Hence Φ defines the universal frame class, which falls within the "trivial" category of Conjecture 10.18. The counterexample thus refutes the full guarded collapse but is consistent with the unimodal collapse conjecture below: it shows that the universal frame class is FPU-characterisable by an unguarded-looking but in fact guarded and effective scheme.

Remark 10.8 (Refined open question)*.* The counterexample prompts a refined open question in place of the refuted conjecture: *which* guarded formulas Φ satisfy $\{F : F \models \mathrm{FPU}(\Phi)\} = \mathrm{CWF}$? The verified positive cases are $\Diamond p$, $\Box p$, $\Diamond^k p$, $\Box^k p$, and the uniform-modality positive guarded formulas (Theorem 10.3). The counterexample shows that arbitrary guarded mixtures of $\Diamond$ and $\neg\Diamond$ need not belong to this class. A precise syntactic characterisation of the CWF-classifying guarded fragment is an open problem; we formulate it in Section 12.

10.4. **Outside the guarded fragment: unimodal collapse patterns.** The picture outside the guarded fragment, within the unimodal single-equation setting, turns out to be more restrictive than one might initially expect. We record several representative cases here; a broader list appears in Section 10.6.

The degenerate schemes. The unguarded schemes $p \leftrightarrow p$ and $p \leftrightarrow \neg p$ induce, respectively, the identity and complement operators. $\mathrm{FPU}(p)$ holds iff $|\mathcal{P}(W)| = 1$ iff $W = \emptyset$. $\mathrm{FPU}(\neg p)$ likewise holds iff $W = \emptyset$ (on the empty frame the unique solution is $\emptyset$; on any nonempty frame no solution exists). Both are trivial.

Mixed monotone occurrences. Consider $\Phi = p \wedge \Diamond p$. The operator T_Φ is monotone; fixed points are the sets X in which every $w \in X$ has some successor in X. The empty set is always a fixed point; any nonempty fixed point must contain an infinite R-forward chain. Hence $\mathrm{FPU}(p \wedge \Diamond p) \iff F$ has no infinite R-forward chain starting from any world, $\iff F$ is converse well-founded. An unguarded scheme captures the same property as the guarded collapse.

First-order traces via $\Diamond\top$-parameters. Replacing p by a $\Diamond$-parameter yields first-order conditions. For instance: $\mathrm{FPU}(p \vee \Diamond\top) \iff R$ is serial; $\mathrm{FPU}(p \wedge \Diamond\top) \iff R = \emptyset$; $\mathrm{FPU}(p \wedge \Diamond\Diamond\top) \iff R^2 = \emptyset$. These are all first-order.

A parametric scheme with an unguarded branch. Consider

$$\Phi_\omega(p, q) \;=\; (q \wedge \Diamond p) \vee (\neg q \wedge p).$$

The occurrence of p on the right-hand disjunct is unguarded. Under the valuation $V(q) = W$, Φ_ω reduces to $\Diamond p$ and FPU requires CWF; under $V(q) = \emptyset$, Φ_ω reduces to p and FPU requires $W = \emptyset$. Since FPU demands uniqueness under *all* valuations, $\mathrm{FPU}(\Phi_\omega) \iff W = \emptyset$. A typical pattern: unguarded occurrences, when activated by a parameter choice, force trivialisation.

The cumulative pattern—every example collapsing into FO, into CWF, or into triviality—motivates Conjecture 10.18 below. The genuine expressive richness of FPU, we shall argue, appears only under signature extension.

10.5. **The classification question and the MSO bound.** The preceding analysis motivates a general classification problem.

Problem 10.9 (The FPU-spectrum)**.** For each modal fragment $\mathcal{F}$ of the propositional modal language, determine the collection

$$\mathrm{Spec}_{\mathrm{FPU}}(\mathcal{F}) \;=\; \{\mathcal{K} \subseteq \mathbf{KFr} : \exists \Phi \in \mathcal{F},\, \forall F \in \mathbf{KFr},\; F \in \mathcal{K} \iff F \models \mathrm{FPU}(\Phi)\},$$

where $\mathbf{KFr}$ is the class of all Kripke frames and a frame class $\mathcal{K}$ is FPU-*characterisable in* $\mathcal{F}$ iff it belongs to $\mathrm{Spec}_{\mathrm{FPU}}(\mathcal{F})$.

Theorem 10.3 shows that CWF (plus trivial cases) belongs to $\mathrm{Spec}_{\mathrm{FPU}}(\text{Guarded})$, at least for the syntactic forms (a)–(c). Proposition 10.5, however, shows that $\mathrm{Spec}_{\mathrm{FPU}}(\text{Guarded})$ contains more than just $\{\mathrm{CWF}\} \cup \{\text{trivial}\}$: the universal frame class $\mathbf{KFr}$ is also FPU-characterisable by a guarded effective formula. The precise structure of $\mathrm{Spec}_{\mathrm{FPU}}(\text{Guarded})$ is open. The first question we address here is whether the full modal language provides a spectrum strictly richer than the first-order one. We answer this in two steps: we first establish an unconditional upper bound, then a witness of strictness in the polymodal extension.

10.5.1. *The MSO upper bound.* The classical standard translation provides a direct route to an unconditional upper bound.

Lemma 10.10 (Standard translation)**.** *There is a translation* ST *assigning to each modal formula φ in variables $p_1, \ldots, p_n$ a first-order formula $\mathrm{ST}(\varphi)(x, P_1, \ldots, P_n)$ with one free individual variable x and n free monadic predicates P_i, such that for every frame $F = (W, R)$, every valuation V, and every world $w \in W$,*

$$(F, V, w) \models \varphi \iff F \models \mathrm{ST}(\varphi)(w)[P_i / V(p_i)].$$

Proof. Standard; see [5, Proposition 2.47]. The clauses are $\mathrm{ST}(p_i) = P_i(x)$, $\mathrm{ST}(\bot) = \bot$, $\mathrm{ST}(\neg\psi) = \neg\mathrm{ST}(\psi)$, $\mathrm{ST}(\psi \wedge \chi) = \mathrm{ST}(\psi) \wedge \mathrm{ST}(\chi)$, $\mathrm{ST}(\Box\psi)(x) = \forall y\,(xRy \to \mathrm{ST}(\psi)(y))$, and analogously for $\Diamond$. □

Theorem 10.11 (MSO boundedness of FPU)**.** *For every modal formula $\Phi(p, q_1, \ldots, q_n)$, the frame class $\{F : F \models \mathrm{FPU}(\Phi)\}$ is monadic-second-order definable.*

Proof. Let $\psi(x, P, Q_1, \ldots, Q_n) := \mathrm{ST}(\Phi)(x)$, where P translates p and Q_i translates q_i. For a subset $X \subseteq W$ and a tuple $\vec{Q} = (Q_1, \ldots, Q_n) \in \mathcal{P}(W)^n$, the predicate "$X$ is a semantic solution of $p \leftrightarrow \Phi$ under $V[\vec{q} := \vec{Q}]$" is first-order expressible (with X appearing as a free monadic predicate in place of P) as

$$\sigma(X, \vec{Q}) \;:=\; \forall x \left(X(x) \leftrightarrow \psi(x, X, \vec{Q})\right).$$

The FPU-condition unpacks as existence plus necessary coincidence:

$$\mathrm{FPU}_\Phi \;:=\; \forall Q_1 \cdots \forall Q_n \Big[\big(\exists X\, \sigma(X, \vec{Q})\big) \;\wedge\; \big(\forall X \forall Y (\sigma(X, \vec{Q}) \wedge \sigma(Y, \vec{Q}) \rightarrow X = Y)\big) \Big].$$

Every quantifier in FPU_Φ ranges over monadic subsets of W (the Q_i, X, Y) or over worlds, and every atomic predicate is either R, equality, or a set-membership instance. Hence FPU_Φ is a monadic second-order sentence in the vocabulary $\{R, =\}$, and $F \models \mathrm{FPU}(\Phi) \iff F \models \mathrm{FPU}_\Phi$. □

Remark 10.12. Theorem 10.11 is uniform in Φ: the quantifier prefix of FPU_Φ has the shape $\forall \vec{Q}\, [(\exists X \forall x) \wedge (\forall X \forall Y \forall x)]$. This places every FPU_Φ at the $\forall^1 \exists^1$ level of the monadic second-order quantifier hierarchy (a universal second-order prefix followed by an inner existential over monadic sets), giving the FPU-spectrum an MSO embedding of low second-order complexity.

10.5.2. *Non-first-order* FPU-*characterisability in the polymodal extension.* Theorem 10.11 is of interest only if the MSO bound is not trivially realised by FO-definable classes. We establish non-triviality in the polymodal extension of the framework.

Fix a polymodal signature $\{R_1, \ldots, R_k\}$ with modalities $\Box_i, \Diamond_i$ associated with R_i. The definitions of guardedness, induced operator, and FPU extend verbatim. The companion result of Theorem 5.1 for systems (Theorem 9.4) applies componentwise.

Theorem 10.13 (Non-first-order FPU-characterisability)**.** *In the bimodal signature* $\{R_1, R_2\}$*, the frame class*

$$\mathcal{K}_2 \;=\; \{(W, R_1, R_2) : R_1 \text{ and } R_2 \text{ are both converse well-founded}\}$$

is FPU-*characterisable (by a polymodal system of guarded equations), is* MSO-*definable, is* not *first-order definable, and is distinct from converse well-foundedness of either single relation.*

Proof. FPU-*characterisability.* Consider the system

$$p_1 \leftrightarrow \Diamond_1 p_1, \qquad p_2 \leftrightarrow \Diamond_2 p_2.$$

The system is *decoupled*: equation i involves only p_i and R_i. A solution is a pair $(X_1, X_2) \in \mathcal{P}(W)^2$ with $X_i = T_{\Diamond_i p_i}(X_i)$ for $i = 1, 2$. By decoupling, uniqueness of the pair (X_1, X_2) is equivalent to uniqueness of each X_i in its respective unimodal reduct. Applying Theorem 6.4 to each reduct separately, uniqueness holds for both components iff both R_1 and R_2 are converse well-founded, i.e. iff $F \in \mathcal{K}_2$.

Note: this part of the argument invokes the unimodal characterization (Theorem 6.4) twice, not the vectorial uniqueness theorem (Theorem 9.4); the decoupled system requires no coupled analysis.

MSO-*definability.* By Theorem 10.11 extended to polymodal vocabularies.

Non-first-order definability. Assume, toward contradiction, that $\mathcal{K}_2$ is defined by a first-order sentence χ over $\{R_1, R_2\}$. For each $n \in \mathbb{N}$, let $F_n = (\{0, 1, \ldots, n\}, R_1^{(n)}, \emptyset)$ with $R_1^{(n)} = \{(i, i+1) : 0 \le i < n\}$. Both relations are converse well-founded (the second trivially, the first as a finite acyclic chain), so $F_n \in \mathcal{K}_2$ and $F_n \models \chi$. Let $F^* = \prod_n F_n / \mathcal{U}$ for a nonprincipal ultrafilter $\mathcal{U}$. By Łoś's theorem, $F^* \models \chi$, hence $F^* \in \mathcal{K}_2$, so R_1^* is converse well-founded. However, fixing representatives $a_k^{(n)} = \min(k, n)$, the set $\{n : a_k^{(n)}\, R_1^{(n)}\, a_{k+1}^{(n)}\}$ contains $\{n \ge k+1\}$, which is $\mathcal{U}$-large; Łoś then gives $[a_k^{(n)}]\, R_1^*\, [a_{k+1}^{(n)}]$ for all $k \ge 0$, exhibiting an infinite forward R_1^*-chain, a contradiction.

Distinctness from single-relation CWF. The frame $G = (\{0, 1, 2, \dots\}, \emptyset, \{(i, i+1) : i \geq 0\})$ has $R_1 = \emptyset$ trivially converse well-founded but R_2 is the non-CWF successor relation. Hence G belongs to $\{F : R_1 \text{ is CWF}\}$ but not to $\mathcal{K}_2$; by symmetry, the same holds for $\{F : R_2 \text{ is CWF}\}$. Hence $\mathcal{K}_2$ is strictly contained in—and therefore distinct from—each single-relation CWF-class. □

Remark 10.14 (Scope of the distinction). The non-first-order-definability of $\mathcal{K}_2$ is inherited from the unimodal case: already CWF itself is not FO-definable by a standard compactness argument, and this property transfers to the bimodal setting. The substantive content of Theorem 10.13 is therefore not merely "FPU-characterisation escapes first-order logic" (already witnessed unimodally by Theorem 6.4), but rather: (i) the *conjunction* $\mathcal{K}_2 = \mathrm{CWF}(R_1) \cap \mathrm{CWF}(R_2)$ is FPU-characterisable by a *system* of guarded equations; (ii) it is strictly distinct from CWF of either single relation; (iii) no single bimodal formula is known to characterise $\mathcal{K}_2$ via FPU (for example, the natural candidate $\Diamond_1 p \vee \Diamond_2 p$ gives FPU $\iff R_1 \cup R_2$ is CWF, a strictly stronger condition than $\mathcal{K}_2$). The necessity of systems for capturing $\mathcal{K}_2$ is what distinguishes the polymodal case from the unimodal single-equation case.

Remark 10.15. The argument generalises. For any $k \geq 1$, the class of k-modal frames in which all k relations are converse well-founded is FPU-characterisable (by the natural k-equation system), MSO-definable in the k-modal vocabulary, and non-FO-definable. The classes for different k live in distinct vocabularies and cannot be directly compared as frame classes; the comparison holds *within a fixed vocabulary* after projecting out redundant relations.

10.6. **The unimodal single-equation landscape.** The polymodal argument of Theorem 10.13 exploits the availability of two independent accessibility relations. One might expect the same expressive gain to be available in the unimodal single-equation setting. Available evidence suggests otherwise.

For every guarded unimodal formula $\Phi(p)$ covered by Theorem 10.3 (forms (a)–(c)) with p effective on F, $\mathrm{FPU}(\Phi) \iff \mathrm{CWF}$. However, Proposition 10.5 shows that there exist other guarded effective formulas whose FPU-class is not CWF but the universal frame class: the guarded fragment is therefore *not* uniformly CWF-characterising. A refined question is which additional frame classes beyond $\{\mathrm{CWF}, \mathrm{universal}, \mathrm{empty}\}$ appear in $\mathrm{Spec}_{\mathrm{FPU}}(\mathrm{Guarded})$. For unguarded unimodal single-equation schemes, every case we have examined falls into one of four categories, which we record here.

- $p \leftrightarrow p$: FPU $\iff W = \emptyset$ (trivial).
- $p \leftrightarrow \neg p$: FPU $\iff W = \emptyset$ (trivial; on empty W the unique solution is $\emptyset$, otherwise no solution exists).
- $p \leftrightarrow p \vee \Diamond\top$: FPU $\iff R$ is serial (FO).
- $p \leftrightarrow p \wedge \Diamond\top$: FPU $\iff R = \emptyset$ (FO).
- $p \leftrightarrow p \wedge \Diamond\Diamond\top$: FPU $\iff R^2 = \emptyset$ (FO).
- $p \leftrightarrow p \wedge \Diamond p$: FPU $\iff$ CWF. (*Unguarded scheme capturing* CWF: see Remark 10.16.)
- $p \leftrightarrow \Box p \to p$: FPU $\iff R$ is reflexive (FO). Verification: a set X is a fixed point iff every $w \notin X$ has all successors in X; equivalently, $W \setminus X$ is R-independent (no R-edge within $W \setminus X$). $X = W$ is always a fixed point. Unique iff no nonempty R-independent subset exists, iff every w has wRw (reflexivity); under reflexivity, any nonempty Y contains some w with $(w, w) \in (Y \times Y) \cap R$, hence fails R-independence.
- $p \leftrightarrow \Diamond p \to p$: FPU $\iff R \subseteq \Delta$ (every edge is a self-loop; FO). Verification: X is a fixed point iff every $w \notin X$ has some successor in X; $X = W$ is always a fixed point. If wRv with $w \neq v$, then $Y = \{w\}$ gives a second fixed point $X = W \setminus \{w\}$; conversely, if $R \subseteq \Delta$, for any nonempty Y every $w \in Y$ has successors only among $\{w\} \subseteq Y$ or none, hence not "outbound", forcing $X = W$ uniquely.

Remark 10.16. The entry $p \leftrightarrow p \wedge \Diamond p$ above is not guarded in p (the left occurrence is unguarded), yet it also characterises CWF via FPU. This shows that CWF is FPU-characterisable by *multiple* syntactically distinct schemes, not only by guarded ones. The narrative is therefore not "the guarded fragment uniquely captures CWF" but rather: "the guarded fragment captures CWF *uniformly* (Theorem 10.3), and certain specific unguarded schemes do so as well."

Remark 10.17. The corrected identifications for $p \leftrightarrow \Box p \to p$ and $p \leftrightarrow \Diamond p \to p$ reveal a structural pattern. Both schemes have the form $Mp \to p$ where Mp is guarded but the outer p is unguarded. Such "antecedent-guarded" schemes appear to characterise first-order properties of R rather than CWF. A systematic investigation of the FPU-classes of schemes $Mp \to p$ (for varying modal prefixes M) is an interesting companion question to Conjecture 10.18.

Every example falls into $\mathrm{FO} \cup \{\mathrm{CWF}, \mathrm{trivial}\}$. This suggests the following refined conjecture, in a direction *opposite* to what the polymodal result might initially have suggested.

Conjecture 10.18 (Unimodal FPU-collapse)**.** *For every modal formula $\Phi(p, \vec{q})$ in the unimodal signature $\{R\}$, the class $\{F : F \models \mathrm{FPU}(\Phi)\}$ is either first-order definable, equal to* CWF*, or trivial (empty or universal frame class).*

The structural mechanism behind this collapse is the following. The FPU-condition is a strong uniformity constraint: it requires uniqueness across *all* valuations of parameters. Any non-trivial freedom in the induced operator under some valuation tends to produce multiple fixed points, forcing FPU to fail. The remaining schemes either (a) have operators that collapse to constants (trivial FPU), (b) have operators whose non-monotone behaviour is rigidly controlled by the modal delay (guarded case, collapsing to CWF), or (c) have operators whose fixed-point structure reflects a simple first-order condition on R. A proof of Conjecture 10.18 would require a syntactic classification of unimodal modal formulas modulo FPU-equivalence. We leave this as an open problem.

Remark 10.19. If Conjecture 10.18 is correct, then the genuine expressive gain of FPU over first-order correspondence arises from *signature extension*—polymodal, vectorial, or parametric—rather than from the intrinsic fixed-point-uniqueness mechanism acting within the unimodal single-equation language. This is an instructive negative structural observation: the polymodal case (Theorem 10.13) is not merely the simplest witness but, conjecturally, the *first* non-trivial witness.

10.7. **Relative position among modes of frame characterization.** We summarise the landscape in light of the preceding results.

Signature / fragment	FPU-**spectrum**	**Status**
Unimodal, guarded single eq. (forms (a)–(c))	$\ni$ CWF	Theorem 10.3
Unimodal, guarded single eq. (general)	$\ni$ CWF, $\ni$ universal	Thm. 10.3 + Prop. 10.5
Unimodal, arbitrary single eq.	$\subseteq \mathrm{FO} \cup \{\mathrm{CWF}, \mathrm{trivial}\}$	Conj. 10.18
Polymodal, systems	$\supsetneq$ FO; $\ni \bigcap_i \mathrm{CWF}(R_i)$	Thm. 10.13
Any signature, any Φ	$\subseteq$ MSO	Theorem 10.11

Mode of characterization	**Condition on F**	**Example**
Classical correspondence	$F \models \varphi$	$F \models \Box p \to p \iff$ reflexive
MSO-definability	$F \models \sigma$ (2nd-order)	CWF via $\forall P(\dots)$
FPU (this paper)	$\lvert\mathrm{Fix}(T_\Phi^V)\rvert = 1$ for all V	$\mathrm{FPU}(\Diamond p) \iff$ CWF
¬FPE (non-existence)	$\lvert\mathrm{Fix}(T_\Phi^V)\rvert = 0$ for some V	Prop. 9.9 (on C_2)
μ/ν coincidence	$\mu p.A = \nu p.A$ for positive A	Arnold–Niwiński [3]

The FPU-mode is distinct from classical correspondence (validity vs. uniqueness of solutions) and, for non-monotone operators, from μ/ν coincidence (Proposition 8.2). For positive monotone formulas, FPU and μ/ν coincidence are equivalent (the proof of Proposition 8.2 shows that (ii) implies (i); the converse for monotone operators follows from Knaster–Tarski, since a singleton Knaster–Tarski interval forces uniqueness across all monotone fixed points, but does not force uniqueness for non-monotone operators). The interesting separation therefore concerns the non-monotone case. FPU is bounded above by MSO-definability unconditionally (Theorem 10.11), and the bound is strict in the polymodal setting (Theorem 10.13). The precise structure of the unimodal spectrum (Conjecture 10.18) remains open.

11. Rank-Locality: The Semantic Core

The uniqueness theorem (Theorem 5.1) has been presented as a consequence of the syntactic condition of guardedness together with converse well-foundedness. In this section we isolate the *semantic* content of the argument and clarify how guardedness functions merely as one syntactic implementation of a more general structural principle.

11.1. **The semantic principle.**

Definition 11.1. An operator $T\colon \mathcal{P}(W) \to \mathcal{P}(W)$ is *rank-local* on a converse well-founded frame F if for every ordinal α and every $w \in W_\alpha$, the value $(T(X))(w)$ depends only on $X \cap W_{<\alpha}$. Equivalently: if $X \cap W_{<\alpha} = Y \cap W_{<\alpha}$, then $T(X)$ and $T(Y)$ agree at every $w \in W_\alpha$.

Theorem 11.2 (Rank-locality implies uniqueness)**.** *Let F be a converse well-founded Kripke frame and $T\colon \mathcal{P}(W) \to \mathcal{P}(W)$ a rank-local operator. Then T has a unique fixed point in $\mathcal{P}(W)$.*

Proof. The proof is identical to that of Theorem 5.1, replacing every appeal to guardedness through Lemma 4.1 by a direct appeal to rank-locality. Existence: construct X by transfinite recursion on rank, defining X_α at each $w \in W_\alpha$ using T applied to any extension of $X_{<\alpha}$; rank-locality ensures the extension is immaterial. Uniqueness: if X, Y are fixed points, transfinite induction on α shows they agree on $W_{<\alpha}$, with the induction step using rank-locality to transfer agreement from $W_{<\alpha}$ to W_α. □

11.2. **Guardedness as a syntactic implementation.**

Proposition 11.3. *Let F be converse well-founded and $A(p)$ guarded in p. Then T_A is rank-local.*

Proof. Let $w \in W_\alpha$ and assume $X \cap W_{<\alpha} = Y \cap W_{<\alpha}$. By Lemma 3.3(b), $R^+(w) \subseteq W_{<\alpha}$, so X and Y agree on $R^+(w)$. Lemma 4.1(b) then gives $(F, V[p := X], w) \models A \iff (F, V[p := Y], w) \models A$, i.e. $(T_A(X))(w) = (T_A(Y))(w)$. □

Corollary 11.4. *If T is rank-local on a* CWF *frame F, $X \neq Y$, and α is the least ordinal with $X \cap W_\alpha \neq Y \cap W_\alpha$, then $T(X)$ and $T(Y)$ agree on $W_{\leq\alpha}$; the first point of disagreement of $T(X)$ and $T(Y)$ has rank strictly greater than α.*

Proof. For w of rank $< \alpha$, X and Y agree on $W_{<\rho(w)}$ by minimality of α, so by rank-locality $T(X)(w) = T(Y)(w)$. For w of rank α, X and Y agree on $W_{<\alpha}$ by minimality of α, so by rank-locality again $T(X)(w) = T(Y)(w)$. □

11.3. **Structural consequences.**

Theorem 11.2 clarifies the deductive flow:

1. **Semantic content**: rank-locality + CWF ⇒ unique fixed point.
2. **Syntactic criterion**: guardedness ⇒ rank-locality (on CWF frames).
3. **Applied theorem**: guardedness + CWF ⇒ unique fixed point.

This separation invites two natural questions not addressed in this paper. First, are there other syntactic criteria beyond guardedness that imply rank-locality? Candidates include: fragments of the modal μ-calculus with guardedness conditions; fragments of first-order modal logic with bounded variable depth; hybrid logic formulas with appropriate nominal usage. A systematic investigation of "rank-local fragments" would generalise the present framework substantially. Second, the converse question: given a rank-local operator T, does there exist a modal formula inducing T? The question is whether rank-locality has a syntactic characterisation via modal formulas, analogous to the characterisation of monotone modal operators as those induced by positive formulas.

Remark 11.5 (Analogy with guarded recursion, refined). Rank-locality is the ordinal-indexed analogue of contractiveness in guarded recursion [4]: a contractive endofunctor on the topos of trees shifts the first point of disagreement strictly forward through the step index; a rank-local operator shifts it through the ordinal rank (Corollary 11.4). Theorem 11.2 is the ordinal analogue of the fixed-point theorem for contractive endofunctors. A formal embedding of converse well-founded Kripke frames with rank-local operators into a topos containing the topos of trees, under which rank-local operators correspond to contractive endofunctors and Theorem 11.2 becomes a corollary of the contractive-endofunctor fixed-point theorem, remains an open problem (Section 12).

12. Discussion and Open Problems

The central contributions of this paper are Theorem 6.4, which characterises converse well-foundedness by a single propositional fixed-point uniqueness condition; Theorem 10.3, which shows that this characterisation extends to a range of guarded syntactic forms (iterated single-modality and uniform-modality positive cases); the counterexample of Proposition 10.5, which delimits the extent of the collapse by exhibiting a guarded effective formula whose FPU-class is the universal frame class rather than CWF; Theorem 10.11, which bounds the entire FPU-spectrum by monadic second-order definability; and Theorem 10.13, which establishes that FPU-characterisation is genuinely distinct from first-order correspondence by exhibiting a non-FO MSO-class in the polymodal extension. Together with the evidence for Conjecture 10.18, these results delineate FPU as a mode of frame characterisation lying strictly between first-order correspondence and MSO-definability, and distinct from μ-calculus collapse theorems.

The following problems are left open.

1. **The CWF-classifying guarded fragment.** By Theorem 10.3, the guarded formulas $\Diamond p$, $\Box p$, $\Diamond^k p$, $\Box^k p$, and uniform-modality positive guarded formulas all satisfy $\mathrm{FPU}(\Phi) \iff \mathrm{CWF}$. By Proposition 10.5, the guarded formula $\Diamond p \wedge \neg\Diamond\Diamond p$ does not. Characterise syntactically the class of guarded formulas Φ for which $\{F : F \models \mathrm{FPU}(\Phi)\} = \mathrm{CWF}$. One natural candidate is the subclass of guarded formulas in which every p-occurrence is positive and all outermost p-guards are of the same modality; this is a proper subclass of the guarded fragment, and the verification of Theorem 10.3 case (c) suggests it as a good starting point.
2. **The unimodal FPU-collapse.** Prove or refute Conjecture 10.18: in the unimodal single-equation setting, every FPU-characterisable class is FO-definable, equal to CWF, or trivial. A proof would require a syntactic classification of unimodal modal formulas modulo FPU-equivalence; a refutation requires an explicit unimodal formula whose FPU-class is neither FO, CWF, nor trivial. Proposition 10.5 provides evidence that the trivial category (in its "universal" form) is non-empty among guarded formulas as well as unguarded ones.
3. **Sahlqvist-like algorithm for FPU.** Classical Sahlqvist theory provides an effective algorithm computing the first-order correspondent of a syntactically restricted modal formula. Is there a Sahlqvist-like procedure that, given a syntactically restricted scheme Φ, outputs a frame-theoretic description of the class $\{F : F \models \mathrm{FPU}(\Phi)\}$ in terms of a MSO-sentence or, where possible, a first-order sentence?
4. **The polymodal FPU-spectrum.** Characterise the set of frame classes FPU-characterisable in polymodal signatures with k relations. Theorem 10.13 gives the class $\bigcap_i \mathrm{CWF}(R_i)$; systematic classification of which k-ary non-FO classes arise is open.
5. **Formal reduction to guarded recursion.** Construct an embedding functor from converse well-founded Kripke frames with guarded operators into a suitable topos

(e.g. $\mathrm{Set}^{\omega^{\mathrm{op}}}$ extended to ordinals [4]), and verify that T_A corresponds to a contractive endofunctor, so that Theorem 5.1 becomes a corollary of the fixed-point theorem for contractive endofunctors.

(6) **Decidability over finite frames.** For a finite frame F and a guarded formula A, computing the unique solution $X = T_A(X)$ is polynomial-time reducible to iterated evaluation by rank. Determine precise complexity bounds for FPU-verification: given F and Φ, decide whether $F \models \mathrm{FPU}(\Phi)$.

(7) **Bisimulation invariance and μ-calculus definability.** By Theorem 10.11, every FPU-characterisable class is MSO-definable. By the Janin–Walukiewicz theorem [10], the bisimulation-invariant fragment of MSO over Kripke structures coincides with the modal μ-calculus. Converse well-foundedness is μ-calculus definable: the least fixed point $\mu p.\Box p$ (iterated from $\emptyset$) yields, at each world, the well-founded core $J = \{w : \text{no infinite } R\text{-chain from } w\}$; consequently, F is converse well-founded iff $\mu p.\Box p$ holds at every world of F, i.e. iff $F \models \mu p.\Box p$ evaluated as a frame property. Hence CWF is bisimulation-invariant, and so is the FPU-class $\{F : F \models \mathrm{FPU}(\Diamond p)\}$ by Theorem 6.4. Similarly, the polymodal class $\mathcal{K}_2$ of Theorem 10.13 is the conjunction of two μ-calculus conditions, hence bisimulation-invariant. The genuine open question is therefore: is every FPU-characterisable class bisimulation-invariant? Equivalently, is every FPU-characterisable class already μ-calculus-definable? This is not guaranteed by MSO-definability alone.

(8) **Non-existence as frame characterization.** Propositions of the form "$p \leftrightarrow \Phi$ has no solution on F" define frame classes via $\neg$FPE (non-existence). Proposition 9.9 exhibits such a class on C_2. Classify frame classes definable this way.

(9) **Temporal and hybrid extensions.** Extend the framework to linear-time and branching-time temporal logics, hybrid logics with nominals, and the guarded fragment of first-order logic.

(10) **Syntactic analogue without transitivity.** Is there a syntactic fixed-point theorem, in the style of de Jongh–Sambin, for converse well-founded frames *without* transitivity, producing a modally definable fixed-point sentence?

The conceptual lesson: semantic determinacy in modal logic can arise from *delayed recursion* rather than from monotonicity. Guardedness delays self-reference; converse well-foundedness prevents indefinite forward unfolding. Together they yield the uniqueness of Theorem 5.1; reciprocally, the frame property is *determined* by such uniqueness (Theorem 6.4), elevating a fixed-point condition to the status of a structural invariant. Within the MSO-ceiling established by Theorem 10.11, the FPU-mode is non-first-order already unimodally: Theorem 6.4 equates $\mathrm{FPU}(\Diamond p)$ with CWF, which is not first-order definable. Polymodal signatures realise further non-first-order frame properties beyond CWF of any single relation (Theorem 10.13), while the unimodal single-equation setting appears—by the evidence of Conjecture 10.18—to enforce a surprisingly rigid collapse into first-order, converse-well-founded, or trivial classes.

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